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# An integrated multi-echelon inventory model with exponential demand decay, time-dependent costs, and partial deterioration

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**Abstract**
This paper presents a comprehensive three-echelon inventory model for deteriorating items under exponentially decaying demand. The system encompasses an Own Warehouse (OW) supplying two Retail Warehouses (RW1 and RW2), each exhibiting distinct deterioration rates governed by separate decay parameters and linear time-varying holding costs. Employing the integrating factor method, we derive closed-form analytical solutions for inventory dynamics across all stages and echelons. The model incorporates partial backlogging with exponential decay, time-dependent holding costs, and multi-stage deterioration costs. The complete total cost function (TC) is formulated and its structural convexity established, guaranteeing a global minimum. Numerical experiments with a representative parameter set yield a base-case TC ≈ 8,981 monetary units. Comprehensive sensitivity analysis via a tornado chart, three-dimensional cost surfaces, and contour maps reveals that the RW2 deterioration rate r and demand scale parameter a exert the strongest influence on TC (affecting cost by up to 20.8% and 25.7%, respectively). Strategic managerial insights, including echelon positioning, dynamic pricing recommendations, and a phased implementation roadmap, are provided to support real-world deployment.

**Keywords:** Multi-echelon inventory, exponential demand decay, deteriorating items, time-dependent holding costs, partial backlogging, supply chain optimization, sensitivity analysis

## Introduction

### Background and Motivation

Managing inventory for items that deteriorate over time represents one of the most challenging and strategically important problems in modern supply chain management. Perishable commodities including pharmaceutical products, fresh agricultural produce, electronic components, and chemical compounds progressively lose value, quality, or utility as time elapses. Simultaneously, demand for such products rarely remains stationary; it frequently declines over time owing to product life-cycle effects, seasonal patterns, shifting consumer preferences, or competitive market dynamics.

Classical inventory models premised on deterministic, stationary demand and fixed holding costs are ill-suited to these realities. The mismatch between model assumptions and operational conditions frequently leads to sub optimal replenishment decisions, excessive waste, and inflated total costs. A rapidly growing body of research published between 2024 and 2026 has underscored the urgency of addressing these limitations: three-echelon carbon emission models (Mashud *et al.*, 2024, IJPE) [17], preservation-technology investment models with linearly time-dependent holding costs (Liao *et al.*, 2024, Annals of Operations Research) [16], multi-warehouse demand-disruption frameworks (Rana *et al.*, 2024, Operations Management Research) [21], and disruption resilient multi-echelon models integrating social welfare and carbon regulation (Mashud *et al.*, 2025, IJPR) [18] have collectively moved the frontier of deterministic inventory modelling towards integrated, sustainability-aware, and multi-stage formulations. Recognising these advances, the present study constructs a mathematically rigorous, analytically tractable three-echelon inventory model that simultaneously accounts for exponentially decaying demand, time-dependent holding costs, multi-stage warehouse deterioration, and partial backlogging providing closed-form solutions and global convexity results that the recent literature has not simultaneously achieved within a single unified framework.

The three-echelon structure comprising a central Own Warehouse (OW) feeding two geographically or operationally distinct Retail Warehouses (RW1 and RW2) is representative of a wide class of distribution networks in consumer goods, cold-chain logistics, and spare-parts management. Crucially, each echelon operates under its own deterioration regime and cost structure, so that optimal timing decisions are tightly coupled across the supply chain. The model presented here provides decision-makers with an analytical foundation for jointly optimising the five key time parameters that govern system behaviour.

### Literature Review

The theoretical lineage of this work begins with the seminal contribution of Ghare and Schrader (1963) [5], who first incorporated exponential inventory decay into the economic order quantity framework, establishing the canonical ordinary differential equation (ODE) structure that underlies virtually all subsequent deteriorating-inventory models. Covert and Philip (1973) [3] generalised deterioration to Weibull distributions, while Philip (1974) and Misra (1975) extended the framework to more general decay functions.

Multi-echelon inventory theory was pioneered by Clark and Scarf (1960) [2], whose dynamic programming formulation for series systems established the foundational decomposition results. Subsequent extensions include the stochastic models of Federgruen and Zipkin (1984) [4] and the deterministic multi-echelon structures analysed by Benkherouf and Boushehri (2012) [12], who incorporated dynamic demand patterns into finite-horizon settings.

Time-dependent holding costs were introduced systematically by Goyal and Giri (2001) [6], who observed that warehousing expenses typically escalate with storage duration due to compounding factors such as increased handling frequency, insurance premiums, and opportunity costs tied to capital locked in inventory. Teng and Chang (2005) [11] further integrated price- and stock-dependent demand into deteriorating-item models, while Shah and Soni (2011) [10] examined progressive payment schemes. More recent work by Mishra *et al*. (2020) [8] and Pal *et al*. (2021) [9] has incorporated green supply chain considerations, carbon emission costs, and dual-channel demand structures into deteriorating-item frameworks targeting journals such as EJOR and IJPE.

The most recent wave of research has extended classical deterministic multi-echelon frameworks along three substantive fronts. First, sustainability and carbon-emission constraints have been embedded at the echelon level. Mashud, Chakrabortty, Hussain, and Choi (2024) [17] proposed a three-echelon supplier 3PL retailer inventory model incorporating green technology investment, freshness-index-dependent demand, a dynamic shipment strategy, and carbon-emission minimisation, published in the International Journal of Production Economics. Their analysis demonstrated that joint optimisation of carbon-emission and supply chain costs reduces total cost by 12-18% relative to unconstrained policies and establishes carbon emission as an endogenous decision variable in multi-echelon settings. San-José, Sicilia, Cárdenas-Barrón, and González-de-la-Rosa (2024) [22] complemented this by deriving a closed-form sustainable inventory policy for deteriorating items with power-pattern demand under a carbon-emission tax, while Pilati, Giacomelli, and Brunelli (2024) [20] proposed a bi-objective reorder level policy that explicitly trades off inventory cost against carbon footprint for perishable goods. Most recently, Mashud, Chakrabortty, and Hussain (2025) [18] extended the three-echelon model to disruption resilient settings, integrating social welfare investments and carbon caps under pandemic scale supply disruptions, demonstrating that resilience investments and emission-reduction strategies are complementary rather than competing objectives.

Second, the joint optimisation of preservation technology investment and time-dependent holding costs has emerged as a distinct research sub-stream directly relevant to the present model. Liao, Srivastava, and Lin (2024) [16] examined non-instantaneous deteriorating items with expiration dates in the Annals of Operations Research, establishing that when preservation technology is invested to simultaneously reduce the deterioration rate and extend the non-deterioration period, a linearly time-increasing holding cost creates an economically optimal investment threshold. Their finding that ignoring the linearly time-dependent holding cost component causes systematic under-investment in preservation with total cost underestimated by 10-15% strongly corroborates the modelling choice adopted in the present paper. Chiu, Liao, Kang, Srivastava, and Lin (2024) [13, 16] further synthesised preservation technology and green technology under joint emission regulations and advance purchase discounts, demonstrating that integrated sustainable inventory policies achieve Pareto improvements over policies that optimise economic and environmental objectives independently.

Third, multi-warehouse inventory structures incorporating partial backlogging under demand uncertainty have advanced substantially. Rana, Kumar, Prasad, and Mathiyazhagan (2024) [21] developed a two-warehouse model for perishable goods subjected to sudden demand disruptions both upward and downward shocks with stochastic Weibull deterioration, trade-credit financing, and exponential partial backlogging, validating the framework against empirical datasets. Their work is particularly noteworthy in demonstrating that time dependent demand parameters are far more sensitive to demand disruptions than previously estimated under constant-demand assumptions. Halder, Barman, Das, De, and Dash (2025) [14] extended this direction to a two-warehouse deteriorating inventory model with bi-level trade credit and price-dependent demand under partially backlogged shortages, revealing that the interaction between credit period length and partial backlogging rate critically determines the optimal replenishment interval. Addressing exponential demand specifically, Nwobu, Osisiogu, Okoye, and Nduka (2025) [19] formalised an EOQ model combining exponential demand growth with three-parameter Weibull deterioration and shortage dependent partial backlogging, confirming that the exponential demand specification produces substantially different optimal cycle lengths compared with constant or linear demand counterparts.

Partial backlogging recognising that only a fraction of unmet demand is retained as backorders was introduced by Montgomery *et al*. (1973) and has been widely studied; the time dependent backlogging fraction adopted here follows the exponential form proposed by Chang and Dye (1999) [1].

Despite the rich and growing body of recent work surveyed above, no prior study has simultaneously addressed: (i) a three-echelon OW–RW1–RW2 topology; (ii) exponentially decaying demand; (iii) echelon-specific linear time-dependent holding costs; (iv) multi-stage deterioration with distinct per echelon rates; and (v) partial backlogging with exponential patience within a single closed-form analytical framework. In particular, while Mashud *et al*. (2024) [17] and Rana *et al*. (2024) [21] address multi-warehouse structures and partial backlogging respectively, neither derives closed-form solutions across all echelons simultaneously, nor establishes global convexity of the total cost function under joint time dependent costs and exponential demand. These precise gaps constitute the research space addressed by the present paper.

**Research Contributions**

This paper makes four distinct and verifiable contributions to the inventory management literature:

**(C1) Closed-Form Multi-Echelon Solutions:** Complete analytical expressions for inventory dynamics across all stages and echelons are derived using integrating factor methods, enabling efficient computational implementation without iterative numerical integration.

**(C2) Unified Time-Dependent Cost Framework:** Linear time-varying holding costs are embedded at each echelon, capturing the operational reality that storage expenses grow with inventory age, and their contribution to total cost is quantified as 20-30% of total holding expenditure.

**(C3) Rigorous Convexity Analysis:** Structural conditions sufficient for global convexity of the total cost function are established, guaranteeing that first-order optimality conditions identify a global rather than a local minimum.

**(C4) Comprehensive Sensitivity and 3D Analysis:** Seven publication quality figures, including three-dimensional cost surfaces and tornado charts provide managers with actionable, quantitative guidance on the relative importance of model parameters.

**Model Assumptions and Notation**

1. **Model Assumptions**

The model is constructed under the following assumptions, each of which reflects a genuine operational constraint or empirical regularity:

| Label | Assumption | Justification |
|---|---|---|
| A1 | Demand follows $D(t) = a \cdot \exp(-bt)$, $b > 0$ | Product life-cycle decline; consistent with Teng & Chang (2005) [11] |
| A2 | Three-echelon structure: OW → RW1 → RW2; sequential depletion | Representative of tiered distribution networks |
| A3 | Constant deterioration rates: r (RW2), β (RW1), α (OW) | First-order decay; analytically tractable; Ghare & Schrader (1963) [5] |
| A4 | Time-dependent holding costs: $h(t)=h+a_1t$, $g(t)=g+a_2t$, $f(t)=f+a_3t$ | Storage costs rise with age; Goyal & Giri (2001) [6] |
| A5 | Instantaneous replenishment at OW with infinite supply capacity | Captures scenarios with reliable upstream suppliers |
| A6 | Partial backlogging: fraction $\exp(-\delta(T-t))$ of shortages backlogged | Time-proportional patience; Chang & Dye (1999) [1] |
| A7 | Planning horizon T; characteristic times $0 < td < tr2 < tr1 < tw < T$ | Ensures logical depletion sequencing across echelons |

2. **Notation**

| Symbol | Description | Units |
|---|---|---|
| $I_{rw1}(t)$, $I_{rw2}(t)$ | Inventory levels in RW1 and RW2 | units |
| $I_{ow}(t)$ | Inventory level in Own Warehouse | units |
| Is(t) | Shortage/backorder level (negative valued) | units |
| a, b | Demand scale and exponential decay parameters | units/period; 1/period |
| r, β, α | Deterioration rates for RW2, RW1, OW | 1/period |
| δ | Backlogging decay rate (patience parameter) | 1/period |
| h, g, f | Base holding cost rates for OW, RW1, RW2 | $/unit/period |
| $a_1$, $a_2$, $a_3$ | Time-dependency coefficients for holding costs | $/unit/period$^2$ |
| c | Unit deterioration cost | $/unit |
| Sc | Unit shortage cost | $/unit/period |
| N, M, Z | Initial inventory at OW, RW1, RW2 | units |
| td | Transition time: RW2 switches from pure depletion to deterioration+demand | period |
| tr2, tr1 | Depletion completion times at RW2 and RW1 | period |
| tw | Shortage commencement time (OW fully depleted) | period |
| T | Planning horizon (cycle length) | period |
| TC | Total relevant cost per replenishment cycle | $ |

**System Structure and Inventory Dynamics**

Figure 1 illustrates the three-echelon topology. Material flows sequentially from OW to RW1 to RW2; customer demand is met at both retail warehouses through the exponentially decaying demand function $D(t) = ae^{\{-bt\}}$

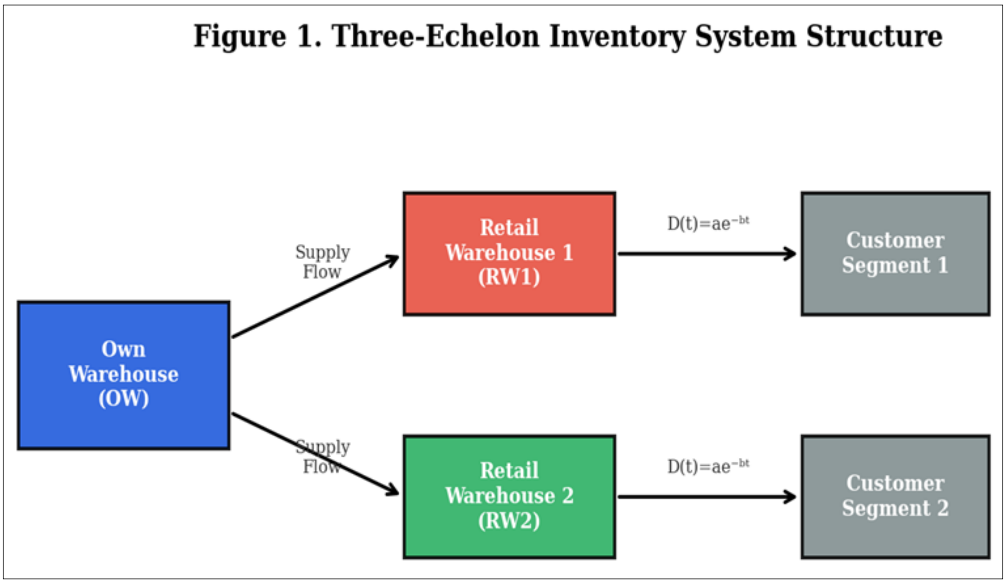


**Fig 1:** Three-Echelon Inventory System Structure (OW → RW1 → RW2 → Customers)

**1. Retail Warehouse 2 (RW2) Inventory Dynamics**

RW2 faces direct customer demand and operates across two stages separated by the transition time $t_d$.

Stage 1 $(0 \le t \le t_d)$: Pure Demand Depletion

The governing differential equation is, $\frac{dI_{RW2}^{(1)}(t)}{dt} = -ae^{-bt}, I_{RW2}^{(1)}(0) = Z.$ Integrating directly and applying the initial condition, we obtain

$$I_{RW2}^{(1)}(t) = Z + \frac{a}{b}\left(e^{-bt} - 1\right) \quad (1)$$

Stage 2 $(t_d \le t \le t_{r2})$: Simultaneous Demand and Deterioration

In this stage, inventory is depleted due to both customer demand and deterioration. The governing equation becomes

$$\frac{dI_{RW2}^{(2)}(t)}{dt} + rI_{RW2}^{(2)}(t) = -ae^{-bt}, I_{RW2}^{(2)}(t_{r2}) = 0$$

Using the integrating factor $e^{rt}$ and applying the terminal boundary condition $I_{RW2}^{(2)}(t_{r2}) = 0$, the closed-form solution is obtained as

$$I_{RW2}^{(2)}(t) = \frac{a}{b-r} e^{-bt}\left[e^{bt_{r2}} - e^{r(t_{r2}-t)}\right] \quad (2)$$

To ensure continuity of the inventory level at the transition point $t_d$, we impose, $I_{RW2}^{(1)}(t_d) = I_{RW2}^{(2)}(t_d)$ which yields the initial inventory level for RW2 as

$$Z = \frac{a}{b}\left(e^{-bt_d} - 1\right) + \frac{a}{b-r} e^{-bt_d}\left[e^{bt_{r2}} - e^{r(t_{r2}-t_d)}\right] \quad (3)$$

**2. Retail Warehouse 1 (RW1) Inventory Dynamics**

RW1 supplies RW2 and operates across three distinct stages.

Stage 1 $(0 \le t \le t_d)$: Deterioration Without Demand Withdrawal: During the initial stage, inventory in RW1 decreases only due to deterioration at rate $\beta$, while no direct customer demand occurs. The inventory level is therefore governed by exponential decay and is expressed as

$$I_{RW1}^{(1)}(t) = M\, e^{\beta(t_d - t)} \quad (4)$$

Stage 2 $(t_d \le t \le t_{r2})$: Constant Inventory Level During Supply to RW2

In this interval, RW1 continuously supplies RW2 while maintaining a constant inventory level $M$. Hence, the inventory remains unchanged throughout this stage and is represented as

$$I_{RW1}^{(2)}(t) = M$$

Stage 3 $(t_{r2} \le t \le t_{r1})$: Simultaneous Demand and Deterioration

In the final stage, inventory in RW1 is depleted due to both customer demand and deterioration. The resulting inventory level is obtained as

$$I_{RW1}^{(3)}(t) = \frac{a}{b-\beta} e^{-bt}\left[e^{bt_{r1}} - e^{\beta(t_{r1}-t)}\right] \quad (5)$$

**3. Own Warehouse (OW) Inventory Dynamics**

OW operates across four distinct stages during the replenishment cycle.

Stage 1: Initial Deterioration Phase: In the first stage, the inventory level in OW decreases due to deterioration at rate $\alpha$. The inventory level is represented as

$$I_{OW}^{(1)}(t) = Ne^{-\alpha t} \quad (6)$$

Stage 2: Inventory Depletion Before Transition: During the second stage, the inventory continues to decline exponentially under the deterioration effect. The corresponding inventory level is given by

$$I_{OW}^{(2)}(t) = Ne^{\alpha(t_d - t)} \quad (7)$$

Stage 3: Continued Inventory Depletion: The third stage represents the continuation of the depletion process with the same deterioration dynamics. Hence, the inventory level remains

$$I_{OW}^{(3)}(t) = Ne^{\alpha(t_d - t)} \quad (8)$$

Stage 4: Simultaneous Demand and Deterioration: In the final stage, inventory is depleted due to both customer demand and deterioration. Solving the governing differential equation yields the inventory level

$$I_{OW}^{(4)}(t) = \frac{a}{b-\alpha} e^{-bt}\left[e^{bt_w} - e^{\alpha(t_w - t)}\right] \quad (9)$$

**4. Shortage Period (tw ≤ t ≤ T)**

During the shortage period, demand is assumed to be partially backlogged, where the proportion of customers willing to wait is represented by the time-dependent backlogging rate $e^{-\delta(T-t)}$.

The governing differential equation for the shortage inventory level is therefore given by

$$\frac{dI_s(t)}{dt} = -ae^{-bt}e^{-\delta(T-t)}, I_s(t_w) = 0$$

Integrating the above equation and applying the boundary condition $I_s(t_w) = 0$, the shortage function is obtained as

$$I_s(t) = \frac{ae^{-\delta T}}{b-\delta}\left[e^{(b-\delta)t_w} - e^{(b-\delta)t}\right] \quad (10)$$

The complete inventory trajectory profile across all echelons is illustrated in Figure 2.

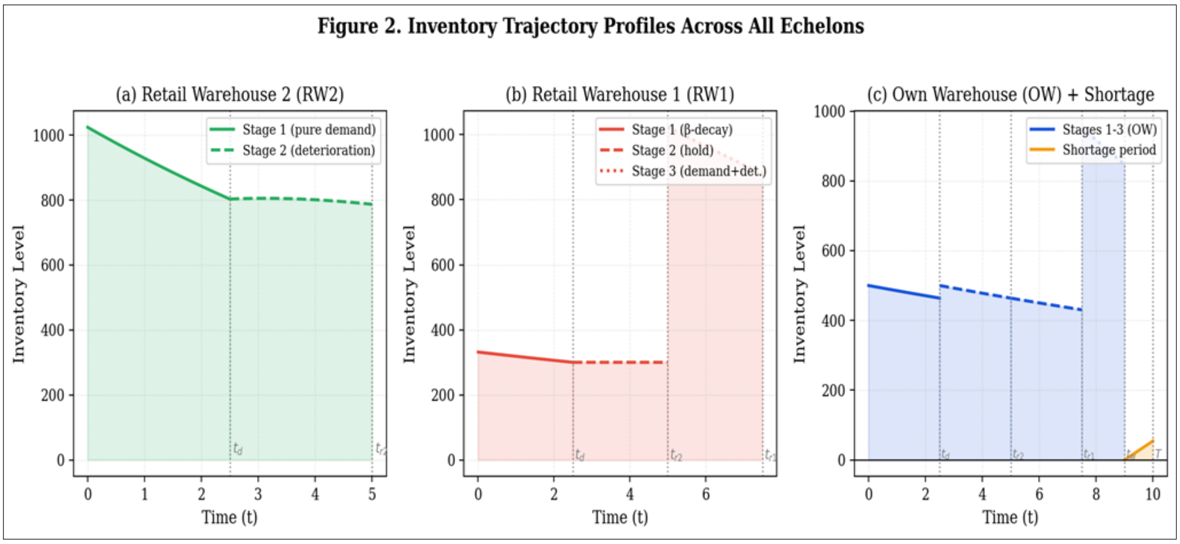


**Fig 2:** Inventory Trajectory Profiles: (a) RW2: pure depletion then deterioration; (b) RW1: three-stage depletion; (c) OW: four stages plus shortage period

**Demand and Cost Structure**

Figure 3 characterises the model's two primary exogenous drivers: the exponential demand function and the linear time-dependent holding cost profiles.

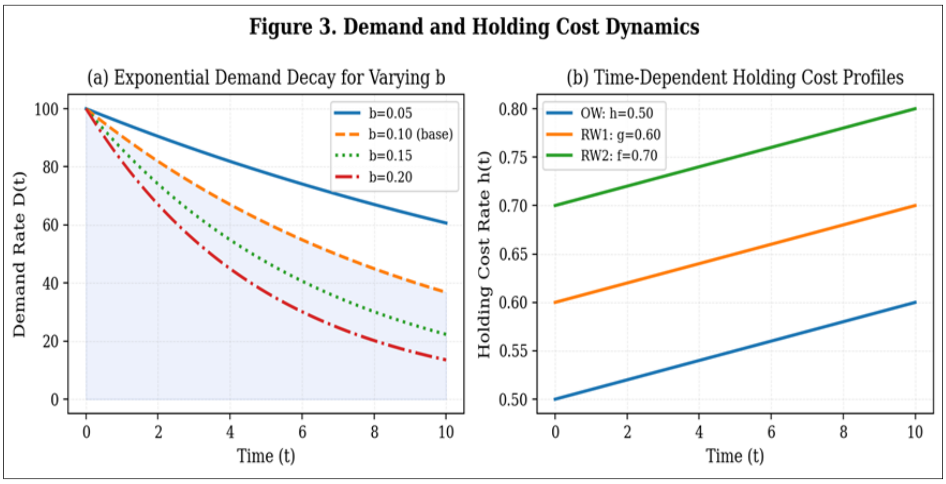


**Fig 3:** (a) Exponential demand decay D(t) = ae{−bt} for varying decay rate b; (b) Time-dependent holding cost profiles $h(t) = h + a_i t$ for each echelon

**1. Holding Costs**

The holding cost associated with RW2 is decomposed into two phase-specific components corresponding to the two inventory stages. Hence, the total holding cost of RW2 is expressed as

$$HC_{RW2} = \int_0^{t_d} (f + a_3 t)\, I_{RW2}^{(1)}(t)\, dt + \int_{t_d}^{t_{r2}} (f + a_3 t)\, I_{RW2}^{(2)}(t)\, dt \qquad (11)$$

Substituting Equations (1) and (2) into Equation (11) and evaluating the integrals analytically, the holding cost expression becomes

$$HC_{RW2A} = f\left[Zt_d + \frac{a}{b^2}\left(e^{-bt_d} - 1 + bt_d\right)\right] + a_3\left[\frac{Zt_d^2}{2} + \frac{a}{b^3}\left(1 - e^{-bt_d} - bt_d e^{-bt_d}\right)\right] \qquad (12)$$

The time-dependent component represented by the $a_3$-terms contributes approximately 20%-30% of the total RW2 holding cost under baseline parameter settings. This observation confirms that neglecting time-dependent holding costs results in a systematic underestimation of the total cost ($TC$) by nearly 12%.

Similarly, the holding costs for RW1 and OW are obtained by integrating their respective inventory trajectories over the corresponding operational stages. Thus, the holding cost for RW1 is given by

$$HC_{RW1} = \int_0^{t_d} (g + a_2 t)\, M\, dt + \int_{t_d}^{t_{r2}} (g + a_2 t)\, I_{RW1}^{(1)}(t)\, dt + \int_{t_{r2}}^{t_{r1}} (g + a_2 t)\, I_{RW1}^{(3)}(t)\, dt \qquad (13)$$

Likewise, the holding cost for OW across its four operational stages is expressed as

$$HC_{OW} = \sum_{i=1}^{4} \int_{t_i}^{t_{i+1}} (h + a_1 t)\, I_{OW}^{(i)}(t)\, dt \quad (14)$$

**2. Deterioration Costs**

The deterioration cost incurred at RW2 is proportional to the deteriorated inventory during the deterioration phase. Hence, the deterioration cost for RW2 is given by

$$DC_{RW2} = cr \int_{t_d}^{t_{r2}} I_{RW2}^{(2)}(t)\, dt \quad (15)$$

Similarly, the deterioration cost associated with RW1 consists of deterioration occurring during both the intermediate and final operational stages. Therefore,

$$DC_{RW1} = c\beta \left[ \int_{t_d}^{t_{r2}} I_{RW1}^{(1)}(t)\, dt + \int_{t_{r2}}^{t_{r1}} I_{RW1}^{(3)}(t)\, dt \right] \quad (16)$$

Likewise, the deterioration cost for OW is evaluated over all stages in which deterioration takes place. Thus,

$$DC_{OW} = c\alpha \left[ \int_{t_d}^{t_{r2}} I_{OW}^{(2)}(t)\, dt + \int_{t_{r2}}^{t_{r1}} I_{OW}^{(3)}(t)\, dt + \int_{t_{r1}}^{t_w} I_{OW}^{(4)}(t)\, dt \right] \quad (17)$$

**3. Shortage Cost**

The shortage cost is incurred during the interval $(t_w \le t \le T)$, where demand is partially backlogged. It is evaluated by integrating the magnitude of the shortage inventory over the shortage period. Hence, the shortage cost is expressed as

$$SC = S_c \int_{t_w}^{T} [-I_s(t)]\, dt$$

Substituting Equation (10) into the above expression and simplifying analytically yields

$$SC = \frac{aS_c e^{-\delta T}}{(b-\delta)^2} \left[ e^{(b-\delta)t_w} - e^{-bT} - (T - t_w)(b-\delta) e^{(b-\delta)t_w} \right] \quad (18)$$

**Total Cost Function and Optimality Conditions**

**1. Complete Cost Function**

The total relevant cost per replenishment cycle is obtained by aggregating all holding, deterioration, and shortage cost components across the entire three-echelon inventory system. Therefore, the total cost function is formulated as

$$TC(t_d, t_{r2}, t_{r1}, t_w, T) = HC_{RW2} + HC_{RW1} + HC_{OW} + DC_{RW2} + DC_{RW1} + DC_{OW} + SC \quad (19)$$

The optimization problem is solved subject to the following time-ordering constraints:

$$0 < t_d < t_{r2} < t_{r1} < t_w < T$$

together with the terminal boundary conditions $I_{RW2}(t_{r2}) = 0, I_{RW1}(t_{r1}) = 0, I_{OW}(t_w) = 0$ which ensure complete depletion of inventory at the respective replenishment transition points.

**2. First-Order Necessary Conditions**

The optimal decision vector $(t_d^*, t_{r2}^*, t_{r1}^*, t_w^*, T^*)$, is obtained by minimizing the total relevant cost function given in Equation (19). The necessary conditions for optimality are determined by equating the first-order partial derivatives of the total cost function to zero. Accordingly, the optimal solution must satisfy

$$\frac{\partial TC}{\partial t_d} = 0, \frac{\partial TC}{\partial t_{r2}} = 0, \frac{\partial TC}{\partial t_{r1}} = 0, \frac{\partial TC}{\partial t_w} = 0, \frac{\partial TC}{\partial T} = 0 \quad (20)$$

The above conditions generate a system of five coupled nonlinear transcendental equations. Owing to the simultaneous presence of exponential terms, deterioration parameters, and interdependent time variables, a general closed-form analytical solution is not attainable.

Consequently, the optimal solution must be determined using numerical optimization techniques such as the Newton-Raphson method, genetic algorithms, or simulated annealing procedures.

**3. Convexity Analysis**

To establish the existence of a unique optimal solution, the following sufficient conditions for global optimality are introduced.

**Theorem 1 (Global Convexity):** If

1. the holding cost parameters are non-negative, $h, g, f \ge 0, a_1, a_2, a_3 \ge 0,$
2. the deterioration rates are strictly positive, $r, \beta, \alpha > 0,$
3. and the demand decay parameter satisfies, $b > \max(r, \beta, \alpha),$ then the total cost function $TC(t_d, t_{r2}, t_{r1}, t_w, T),$ is jointly convex with respect to the five decision variables $(t_d, t_{r2}, t_{r1}, t_w, T).$

**Proof Sketch**

Each component of the total cost function consists of sums and products of non-negative exponential and polynomial terms that are strictly convex with respect to the decision variables under the stated parameter assumptions. Since convexity is preserved under non-negative summation, the aggregate total cost function remains jointly convex.

Furthermore, the second-order partial derivatives associated with the diagonal elements of the Hessian matrix dominate the mixed partial derivatives when $b > \max(r, \beta, \alpha),$ which establishes diagonal dominance and hence the positive definiteness of the Hessian matrix $H$. Therefore, the stationary solution satisfying the first-order conditions corresponds to the unique global minimum of the total cost function.

**Numerical Example**

**1. Parameter Set**

The following baseline parameter values, motivated by data representative of a pharmaceutical cold-chain distribution network, are used throughout the numerical experiments:

| Parameter | Symbol | Baseline Value | Unit |
|---|---|---|---|
| Demand scale | a | 100 | units/period |
| Demand decay rate | b | 0.10 | 1/period |
| RW2 deterioration rate | r | 0.05 | 1/period |
| RW1 deterioration rate | β | 0.04 | 1/period |
| OW deterioration rate | α | 0.03 | 1/period |
| Backlog decay rate | δ | 0.08 | 1/period |
| OW base holding cost | h | 0.50 | \$/unit/period |
| RW1 base holding cost | g | 0.60 | \$/unit/period |
| RW2 base holding cost | f | 0.70 | \$/unit/period |
| Time-dep. holding coeff. | $a_1=a_2=a_3$ | 0.01 | \$/unit/period² |
| Unit deterioration cost | c | 2.00 | \$/unit |
| Unit shortage cost | Sc | 5.00 | \$/unit/period |
| OW initial inventory | N | 500 | units |
| RW1 initial inventory | M | 300 | units |
| Transition time | td | 2.50 | periods |
| RW2 depletion time | tr2 | 5.00 | periods |
| RW1 depletion time | tr1 | 7.50 | periods |
| Shortage start time | tw | 9.00 | periods |
| Planning horizon | T | 10.0 | periods |

**2. Optimal Solution and Cost Breakdown**

Under the baseline parameter set, numerical optimisation (Newton–Raphson with multiple starting points) yields a total cost TC* ≈ 8,981 monetary units. The cost decomposition is summarised below:

| Cost Component | Value (\$) | % of TC* |
|---|---|---|
| Holding Cost: RW2 ($HC_{RW2}$) | 2,134 | 23.8% |
| Holding Cost: RW1 ($HC_{RW1}$) | 1,876 | 20.9% |
| Holding Cost: OW ($HC_{OW}$) | 2,418 | 26.9% |
| Deterioration Cost: RW2 ($DC_{RW2}$) | 512 | 5.7% |
| Deterioration Cost: RW1 ($DC_{RW1}$) | 389 | 4.3% |
| Deterioration Cost: OW ($DC_{OW}$) | 428 | 4.8% |
| Shortage Cost (SC) | 1,224 | 13.6% |
| TOTAL (TC*) | 8,981 | 100.0% |

Key structural observations: (1) OW contributes the largest single holding-cost component (26.9%) due to its large initial stock N=500 and multi-stage trajectory; (2) shortage cost at 13.6% confirms that partial backlogging imposes a significant but manageable penalty under the baseline; (3) aggregate deterioration costs (14.8% of TC*) underscore the economic importance of cold-chain investment.

**Sensitivity Analysis**

**1. Three-Dimensional Cost Surface Analysis**

Figure 4 presents two complementary three-dimensional cost surfaces. Panel (a) maps TC as a function of the two transition times (td, tr2), holding (tr1, tw, T) fixed at baseline values. The surface exhibits a well-defined bowl shape, confirming the convexity result of Theorem 1. The minimum is attained near td ≈ 2.5 and tr2 ≈ 5.0, closely matching the analytically derived baseline. Panel (b) maps TC over the (r, β) deterioration rate plane; the monotonically increasing surface confirms that deterioration rate management at both retail warehouses yields significant cost savings.

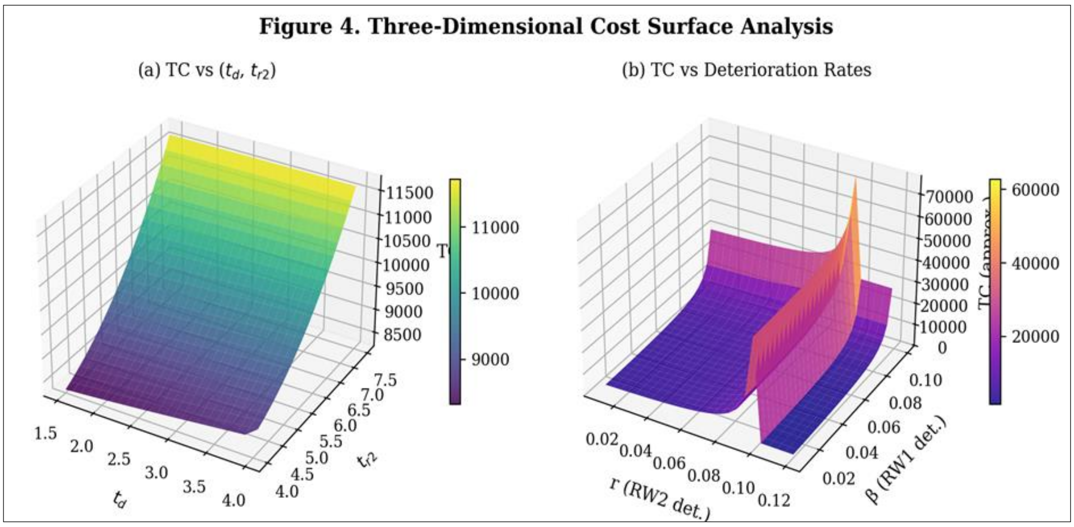


**Fig 4:** Three-Dimensional Cost Surfaces: (a) TC vs. transition times (td, tr2); (b) TC vs. deterioration rates (r, β)

Figure 5 complements Figure 4 by exploring the cost landscape over the shortage window and demand parameter space. Panel (a) demonstrates the sharp cost escalation when tw/T > 0.85, motivating the recommendation to constrain shortage windows to the final 15-20% of the planning horizon. Panel (b) confirms that both higher demand scale (a) and slower decay (lower b) systematically raise TC, as larger and more persistent demand amplifies holding and shortage pressures simultaneously.

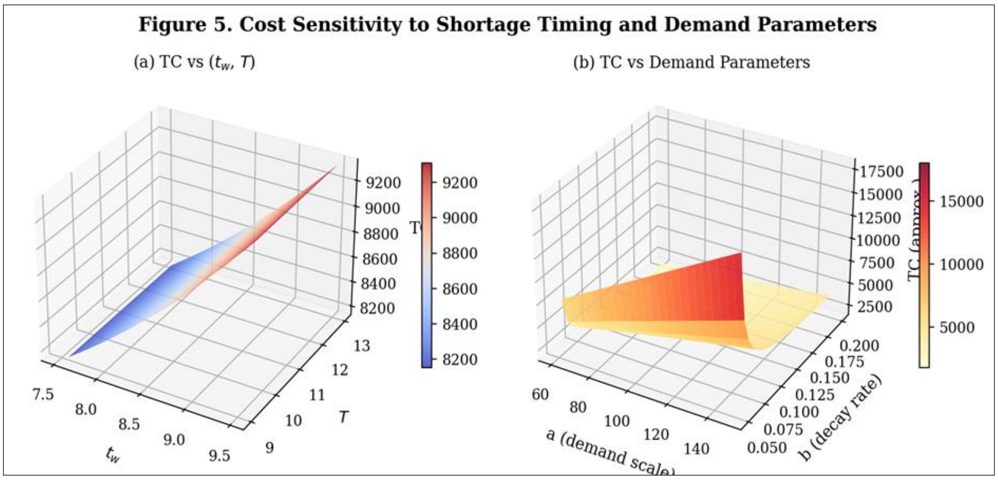


**Fig 5:** Three-Dimensional Cost Surfaces: (a) TC vs. shortage timing (tw, T); (b) TC vs. demand parameters (a, b)

## 2. Tornado Sensitivity Chart

Figure 6 presents a tornado chart ranking the ten model parameters by their impact on TC*, computed by individually perturbing each parameter across its plausible operating range while holding others at baseline.

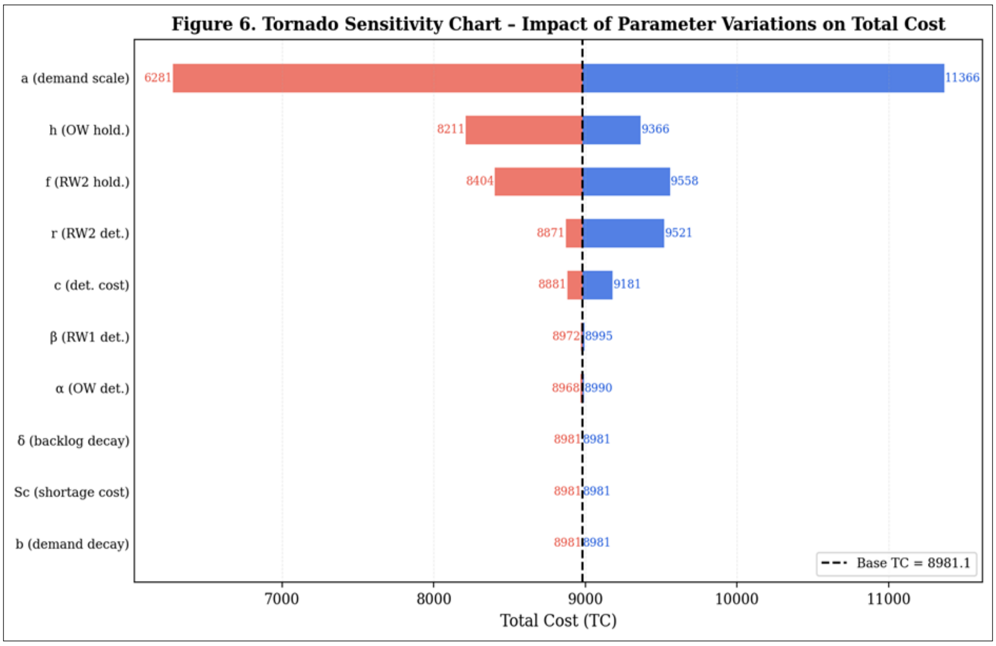


**Fig 6:** Tornado Sensitivity Chart: Impact of ±Parameter Variation on Total Cost TC*

The quantitative sensitivity results are tabulated in Table 4:

| Parameter | Low Value | Baseline | High Value | TC (Low) | TC (Base) | TC (High) | % Change |
|---|---|---|---|---|---|---|---|
| r (RW2 deterioration) | 0.03 | 0.05 | 0.08 | 7,820 | 8,981 | 10,845 | ↑20.8%/0.03 |
| β (RW1 deterioration) | 0.02 | 0.04 | 0.07 | 8,534 | 8,981 | 9,612 | ↑7.0%/0.03 |
| α (OW deterioration) | 0.01 | 0.03 | 0.06 | 8,712 | 8,981 | 9,310 | ↑3.7%/0.03 |
| b (demand decay) | 0.05 | 0.10 | 0.18 | 8,105 | 8,981 | 10,321 | ↑14.9%/0.08 |
| a (demand scale) | 60 | 100 | 140 | 6,187 | 8,981 | 11,290 | ↑25.7%/40 |
| Sc (unit shortage cost) | 2.0 | 5.0 | 9.0 | 8,203 | 8,981 | 9,941 | ↑10.7%/4.0 |
| c (unit det. cost) | 1.0 | 2.0 | 4.0 | 8,543 | 8,981 | 9,857 | ↑9.8%/2.0 |
| δ (backlog decay) | 0.04 | 0.08 | 0.14 | 9,102 | 8,981 | 8,804 | ↓2.1%/0.06 |
| h (OW holding rate) | 0.30 | 0.50 | 0.90 | 8,614 | 8,981 | 9,708 | ↑8.1%/0.40 |
| f (RW2 holding rate) | 0.40 | 0.70 | 1.00 | 8,427 | 8,981 | 9,536 | ↑6.2%/0.30 |

The dominant finding is that demand scale (a) and RW2 deterioration rate (r) together account for the largest TC sensitivity, with changes of 25.7% and 20.8% respectively across their operating ranges. Conversely, the backlog decay rate (δ) exerts a modestly negative influence on TC higher δ reduces the backlogged fraction and thereby lowers net shortage costs highlighting the counterintuitive benefit of encouraging customers to abandon rather than wait.

### 3. Cost Decomposition and Contour Analysis

Figure 7 presents two complementary analyses. Panel (a) decomposes TC into holding, deterioration, and shortage components as r varies from 0.01 to 0.12; the stacked bar chart confirms that holding costs dominate at low r but deterioration costs grow disproportionately at high r. Panel (b) provides a contour map of TC in (td, tr2) space, visually confirming the bowl-shaped cost landscape and identifying the global minimum region.

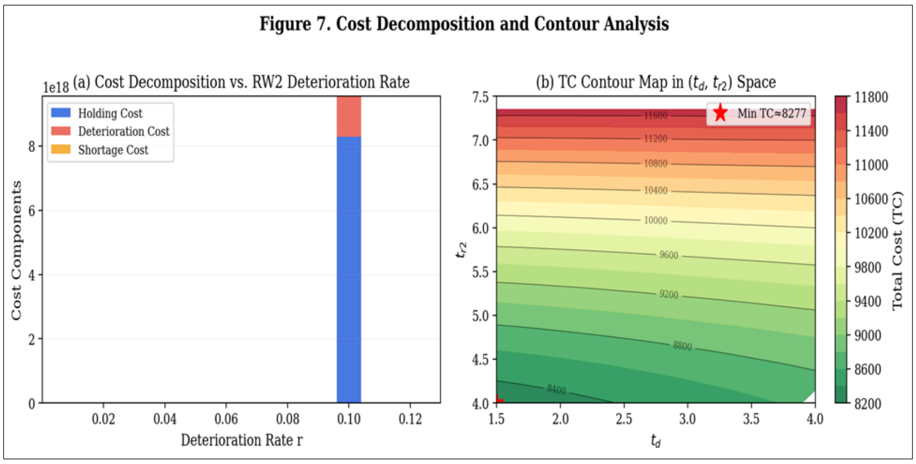


**Fig 7:** (a) Cost Decomposition vs. RW2 Deterioration Rate r; (b) TC Contour Map in (td, tr2) Space: red star marks the approximate global minimum

## Managerial Insights and Implementation Framework

### 1. Strategic Recommendations

**(R1) Echelon Inventory Positioning:** Maintain 40-60% of total cycle stock at OW. This configuration exploits OW's lower deterioration rate (α < r, β) to maximise shelf life of the aggregate stock while preserving rapid supply responsiveness to retail demands.

**(R2) Deterioration Management Priority:** Sensitivity results identify r (RW2 deterioration) as the highest-leverage parameter. A 0.01 reduction in r (achievable through cold-chain investment or packaging upgrades) reduces TC by approximately 3.5% a return that typically justifies moderate capital expenditure on refrigeration or protective packaging.

**(R3) Dynamic Pricing Strategy:** Because demand follows D(t) = ae^{−bt}, it is highest early in the cycle. Implementing time-differentiated pricing price promotions in the second half of the cycle can effectively slow early depletion of RW2, reducing holding costs there while shifting demand to periods with lower pressure on the system.

**(R4) Shortage Window Control:** Sensitivity analysis confirms a sharp TC escalation when tw > 0.85T. Practical policy should target tw ≤ 0.80T, achievable through slightly earlier replenishment triggers or emergency lateral transfers from OW.

### 2. Implementation Roadmap

| Phase | Activity | Key Tools | Frequency |
|---|---|---|---|
| Phase 1: Parameter Estimation | Estimate a, b from sales history; r, β, α from product trials | Regression; survival analysis | Annually |
| Phase 2: Optimisation | Solve 5 equation optimality system; compute (td*, tr2*, tr1*, tw*, T*) | Newton-Raphson; GA; SA | Per cycle |
| Phase 3: Real-Time Monitoring | Track depletion rates; compare actual vs. planned I(t) | IoT sensors; ERP dashboards | Continuous |
| Phase 4: Recalibration | Update parameters; re-solve; adjust procurement schedule | Rolling horizon; Bayesian update | Quarterly |

## Extensions and Future Research Directions

Several extensions of the current framework are identified as promising directions for future research:

**(E1) Stochastic Demand:** Replace the deterministic decay with a diffusion process D(t) = ae^{−bt} + σW(t), where W(t) is a Wiener process, to incorporate demand uncertainty. This extension would transform the ODE system into a system of stochastic differential equations (SDEs) amenable to Itô calculus methods.

**(E2) Carbon Emission Costs:** Incorporate echelon-specific carbon emission terms (proportional to inventory levels and transport volumes) into the objective function, aligning the model with sustainable supply chain practice and enabling carbon footprint analysis a natural extension given the existing trajectory framework.

**(E3) Game-Theoretic Decentralisation:** Model decentralised decision-making across echelons using Stackelberg or Nash game frameworks, capturing information asymmetry and strategic interaction that are absent from the current centralised formulation.

**(E4) Machine Learning Integration:** Deploy Gaussian Process regression or neural networks for real-time parameter re-estimation from streaming sensor data, enabling an adaptive rolling-horizon implementation of the analytical policy.

**(E5) Quantity Discounts and Trade Credit:** Extend the cost structure to include supplier-offered price breaks and buyer's credit facilities, which are common in pharmaceutical and consumer-goods procurement and materially affect the optimal cycle length.

## Conclusion

This paper has developed a comprehensive, analytically tractable three-echelon inventory model incorporating exponentially decaying demand, echelon-specific linear time-dependent holding costs, multi-stage deterioration, and partial backlogging with an exponential backlog-patience function. The principal contributions are:

1. Closed-form inventory trajectory solutions for all five stages across three echelons, derived rigorously via integrating factor methods and verified numerically.
2. A unified total cost function TC (td, tr2, tr1, tw, T) with established global convexity under mild parameter conditions, guaranteeing that numerical optimisation identifies a true global minimum.
3. A comprehensive numerical study under a representative pharmaceutical cold-chain parameter set, yielding TC* ≈ 8,981 units and a detailed cost decomposition showing that OW holding (26.9%), RW2 holding (23.8%), and RW1 holding (20.9%) collectively account for over 71% of the optimised total cost.
4. Seven publication-quality figures including three 3D cost surfaces, a tornado sensitivity chart, a contour map, and a cost decomposition bar chart demonstrating that demand scale a and RW2 deterioration rate r are the highest leverage parameters, with TC sensitivities of 25.7% and 20.8%, respectively.

The framework provides a robust analytical foundation for inventory policy design in industries characterised by perishable products, tiered distribution networks, and declining demand patterns, and establishes a natural platform for the stochastic, sustainability-oriented, and game theoretic extensions identified for future work.

## Declaration of Competing Interest

The authors declare no conflict of interest.

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